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The beauty of discrete mathematics

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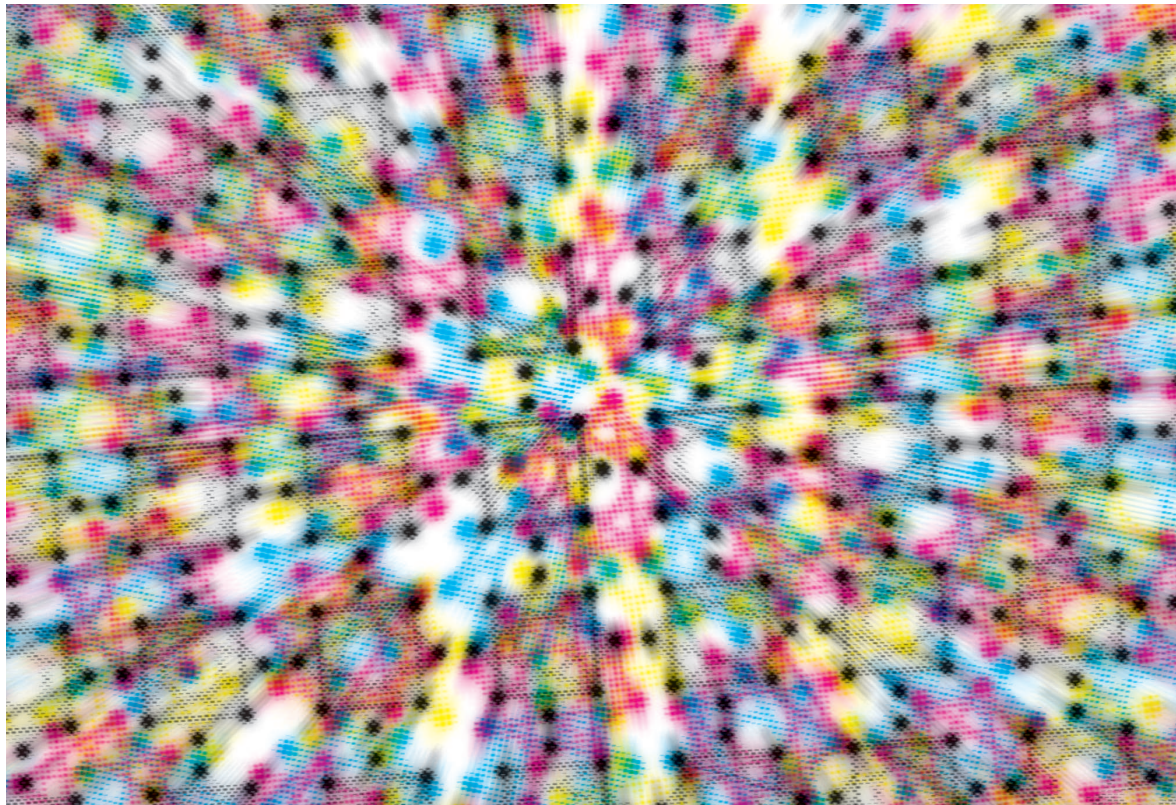
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The beauty of Discrete Mathematics



Imagine you are at a New Year's reception. At such an occasion many hands will be shaken. Some guests will shake only a few hands, whilst others shake the hands of many participants. There is not much structure to be expected in this hand shaking. Yet, something can be said: At the end, there will always be at least two guests who have shaken exactly the same number of hands. TEXT: WILLEM H. HAEMERS

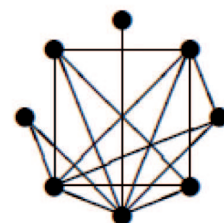
Maybe you think this is a statement about human behavior, but it is not. It is true under all circumstances, for any number of participants greater than one. Even if the guests join their forces in trying to falsify the statement. It is a mathematical theorem and its correctness can be proved. The proof goes by showing that falsity of the statement leads to a contradiction. Let N be the number of participants. The number of hands one participant can shake varies from 0 to $N - 1$ (you don't shake hands with yourself). Assume the statement is false. Then each of the N guests shakes a different number of hands. This means that all numbers from 0 to $N - 1$ do occur. So there is someone who shakes 0 hands, and someone who shakes the hand of all $N - 1$ other guests. The latter person has given a hand to everybody else, including the one who has shaken 0 hands. This is a contradiction. So the assumption must be wrong, and therefore the statement is true.

The theorem just proved, belongs to the Graph Theory, a major part of Discrete Mathematics. A graph consists of a number of nodes, where some pairs of nodes are adjacent. A pair of adjacent nodes is called an edge. In the example of the New Year's reception the nodes are the

guests, and two nodes are adjacent if the guests have shaken hands. With this terminology we can formulate the above claim as follows:

THEOREM 1. In every graph with more than one node, at least two nodes are in the same number of edges.

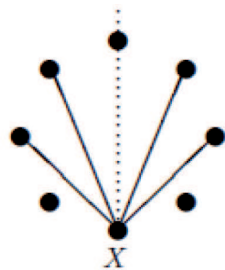
In a drawing of a graph the nodes are represent by dots, and an edge is a line segment between the corresponding two nodes. The figure below gives a drawing of a graph. According to Theorem 1 at least two dots must be endpoint of the same number of line segment (check).



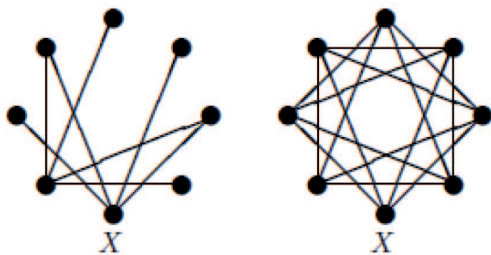
Theorem 1 is a reassuring fact. It states that, within a chaos there is always some order. A different question is, how much order can we construct?

Can, for example, the handshaking be organized such that every guest of the New Year's reception shakes exactly four hands? This is possible and can be realized as follows.

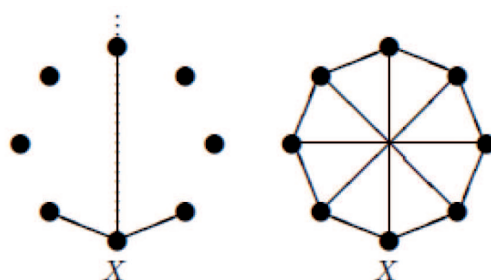
Distribute all nodes equally on a circle. Fix one node, say X , and make X adjacent to four other nodes, but such that the picture is symmetric with respect to the line through X and the center of the circle.



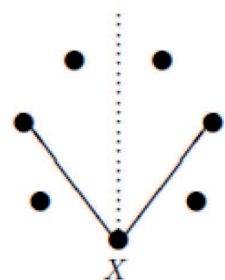
The other edges are obtained by rotating the picture around the center such that each node goes to the next one on the circle. After $N - 1$ such rotation steps all nodes are in the same number of edges, and the required symmetry establishes that rotating introduces no new edges. So at the end all nodes will be in exactly four edges. The picture below gives the situation after one rotation, and after $N - 1$ steps.



Can we replace the number four by any other number K less than N ? In the example with $N = 8$ we can do so. If for example $K = 3$ one can take the left graph in the picture below as starting position. After seven rotation steps we obtain the graph on the right.

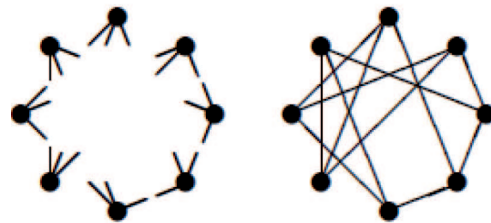


However, if N is odd, the required symmetry is only possible if K is even (see below). So we have a second theorem:



THEOREM 2. If $K < N$, and N or K is even, then there exists a graph with N nodes, such that every node is in exactly K edges.

The described method is not the only one to make graphs with this property. So a natural question is if one of the other methods could give a graph where K and N are both odd. The answer to this question follows by counting edges. Every node has K adjacent nodes. So in total there are $K \times N$ pairs of adjacent nodes. But this way each edge is counted twice (once for each node). Therefore the total number of edges is $(K \times N)/2$, so $K \times N$ is even.



THEOREM 3. If N and K are both odd, there is no graph with N nodes, where every node is in exactly K edges.

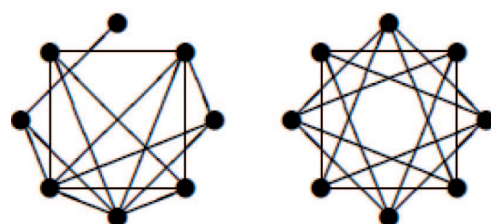
So far we have three types of theorems:

- Theorem 1: Some structure can never be avoided.
- Theorem 2: A required structure can be realized.
- Theorem 3: A required structure is impossible.

These three types of results are typical for Discrete Mathematics.

At this point you may wonder if there are any more serious applications than the counting of shaken hands. Of course there are, but it was my intention to firstly show you the charm and beauty of a mathematical reasoning. Some applications of Discrete Mathematics are:

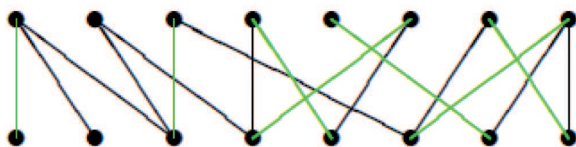
1. **Coding Theory.** The design and analysis of good coding and decoding formulas. As the transport and storage of digital data, the information is encoded. This to detect and correct errors, or to protect information from abuse. Without such protection, electronic financial transactions and commercial wireless communication would be impossible.
2. **Operations Research.** The optimization of business processes. For examples the loading and routing of trucks at a transport company, the assemblage order in a production process, or the making of a lecture schedule for a university.
3. **Statistics.** For several experiments that are carried out according to carefully designed balanced schemes. We illustrate this with an example. A monastery wants to introduce a new kind of beer. The brew master has developed eight candidates. One of these will be chosen. The monks have to decide which candidates tastes best. There are 16 monks. Every monk tastes only two candidates and decides which of the two tastes best. Based on the outcome of the monks, the beer will be chosen.



In this example the schedule for the monks can be represented by a graph. The eight candidates are the nodes, and each monk corresponds to the edge between the two tested candidates. Not every graph with eight nodes and 16 edges provides a useful scheme for this experiment. There are some structure requirements. We will not go into this. But it should be clear that the right graph of the above picture is more suitable than the left one (why?). For this kind of experiments one needs graphs, or schedules that have some kind of regularity. If case beer tasting is not considered a convincing example, we mention that similar experiments are used for testing medicines on people. There it is obviously important to get a best possible result with few subjects. We give one more application.

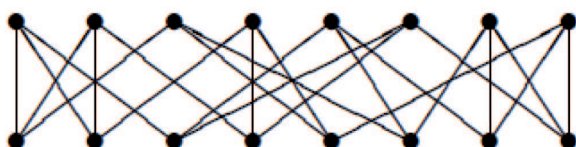
4. Large Networks. The analysis of the structure of very large networks. It turns out that concepts and results from linear algebra are very useful for this application. A famous example is the Page rank, an ordering of the internet pages used by Google which is based on the Perron-Frobenius eigenvector of an underlying matrix, a well-studied algebraic concept. Another important concept is the so-called connectivity of a graph. This is a number (for insiders: the second smallest eigenvalue of the Laplacian matrix) that, thanks to linear algebra, can be computer fast and accurate, even for very many nodes. This number happens to be a good measure for the firmness of the graph. The larger the algebraic connectivity, the more solid is the graph.

As a last example we explain a theorem by the author [1,2] that relates the algebraic connectivity to another structural graph property: the presence of a perfect matching. A collection of edges that have no node in common is called a matching. A matching is perfect if every node is in an edge of the matching. For example, the green edges in the graph below make up a matching, which is not perfect.

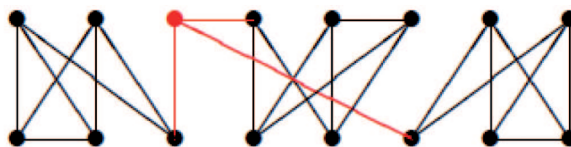


Often such a matching problem is described in terms of relations between boys and girls. In the example the upper nodes are girls and the lower nodes are boys. An edge between a boy and a girl indicates that they can make a good couple. A matching chooses a number of disjoint good couples. The idea is that these good couples get married. With a perfect matching everybody gets married. Sadly this is not possible in the above example; the first two boys both only have the first girls as potential bride. The considered graph is bipartite. This means that all good couples consist of a boy and a girl. In 1916 the Hungarian mathematician Dénes König [3] proved the following:

THEOREM 4. If in a bipartite graph every node is in the same number (not zero of course) of edges then there exists a perfect matching. In the bipartite graph below, every node is in exactly three edges, so a perfect matching exists (find one!).



König lived 100 years ago. In modern times one would also allow good couples between two boys or two girls. The question is: does König's theorem still hold? The answer is given by the graph below.



Every node is in exactly three edges, but the graph is not bipartite. We'll show by contradiction that there is no perfect matching. Assume there exists one, then it clearly contains exactly one of the red edges. Delete the red node and red edges. Then the remaining black graph still has matching with seven couples. But the black graph consists of three disconnected parts with five nodes each. Clearly each part has no more than two couples. Contradiction. The algebraic connectivity of the above graph equals 0.1442 and the next theorem states that this graph is on the borderline for a perfect matching.

THEOREM 5. If every node of a graph is in exactly three edges, and the algebraic connectivity is larger than 0.1442 then the graph has a perfect matching.

For simplicity we restricted to the case of three edges in every node, but a similar result is true for any constant. The threshold value increases with the constant but will always be less than 1. For almost all graphs, however, the algebraic connectivity is much larger than 1. Thus, in a sense, Theorem 5 states that König's theorem remains true for non-bipartite graphs in almost all cases, and that the algebraic connectivity indicates when this can be guaranteed.

This article is based on the inaugural lecture 'Discreet en mooi', given by the author on September 4, 2009 at Tilburg University. See: <http://lyrawww.uvt.nl/~haemers/d+m.pdf>

Endnotes

- [1] A.E. Brouwer & W.H. Haemers, Perfect matchings and eigenvalues, *Linear Algebra Appl.* 395 (2005), 155-162.
- [2] S.M. Cioabă, D.A. Gregory & W.H. Haemers, Matchings in regular graphs from eigenvalues, *J. Combinatorial Theory B* 99 (2009), 287-297.
- [3] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* 77 (1916), 453-465.

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